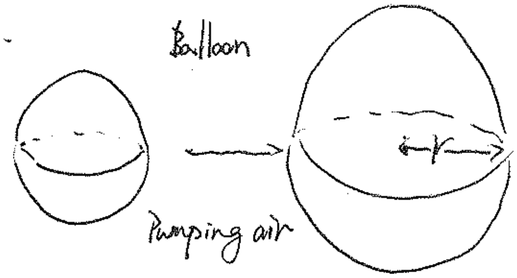
§2.8 Related rates.



$V(t)$: volume at time t

$r(t)$: radius at time t.

V, r is increasing. t

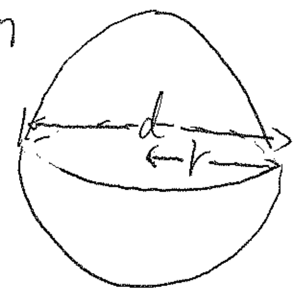
Model: Pumping air into a balloon.

Rate of change of the volume and rate of change of the radius are related to each other. i.e., $V'(t)$ and $r'(t)$ are related to each other.

Question (Goal): What is the exact relation? How to find one given the other rate?

eg. 1. Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\ cm^3/s$. How fast is the radius of the balloon increasing when the diameter is $50\ cm$?

Step 1: Draw a picture with all the given information and relations/formulas.



Known: $V'(t) = 100\ cm^3/s$

diameter $= 50\ cm$.

r = radius $= 25\ cm$.

$d = d(t)$, $r = r(t)$, $V = V(t)$

Step 2:
Given formula: $V = \frac{4}{3}\pi r^3$. $\quad V(t) = \frac{4}{3}\pi \cdot [r(t)]^3$

Step 3: Take derivative with respect to t in the above (step 2) equation

$$V' = (V(t))' = \left(\frac{4}{3}\pi \cdot r^3\right)'$$

$$= \frac{4}{3}\pi \cdot \left[[r(t)]^3 \right]' \overset{\text{chain rule}}{=\!=\!=} \frac{4}{3}\pi \cdot \underbrace{3[r(t)]^2}_{out'(inn)} \cdot \underbrace{r'(t)}_{inn'}$$

outer: $[\blacksquare]^3$

inner: $r(t)$.

i.e. $V'(t) = \frac{4}{3}\pi \cdot r^2(t) \cdot r'(t)$

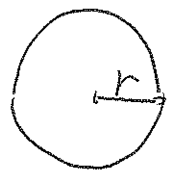
Step 4: Plug in all numerical values (Step 1) and solve for the unknown.

$$100 = \frac{4}{3}\pi \cdot 25^2 \cdot r'(t) \implies r'(t) = \frac{100}{\frac{4}{3}\pi \cdot 25^2} = \frac{3}{25\pi}\ cm/s.$$

Remark: The most important preparation (before taking derivative) is the formula in Step 2. It's direct in some examples (like eg.1). For more complicated questions, you need to find the equation relating the variables according to the geometry of the picture (in Step 1).

e.g. 2 (S16, M-C). If the radius of a circular ink blot is growing at a rate of 3 cm/min. How fast (in cm^2/min) is the area of the blot growing when the radius is 10 cm?

Solution: (Step 1)



Area: $A(t)$

Radius: $r(t)$. $r' = 3$, $r = 10$.

(Step 2) $A = \pi \cdot r^2$

(Step 3) $(A)' = (\pi \cdot r^2)'$ Caution: A, r are both functions of t

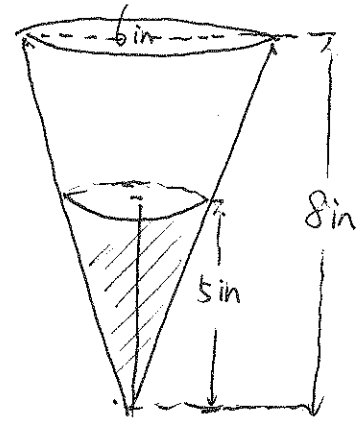
Remember to use chain-rule.

$A'(t) = \pi \cdot (r^2)'$ outer: $\blacksquare^2$ outer$' = 2\blacksquare \xrightarrow[r]{\text{Plug in}} 2 \cdot r$

$= \pi \cdot 2r \cdot r'$ inner: $r(t)$ inn$' = r'$

Step 4 Plug in: $A' = \pi \cdot 2 \cdot 10 \cdot 3 = \boxed{60\pi}$ (cm^2/min)

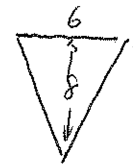
e.g. 3 (Fall 16) (WW#7) A filter filled liquid is in the shape of a vertex-down cone with a height of 8 inches and a diameter of 6 inches at its open (upper) end. If the liquid drips out the bottom of the filter at the constant rate of $7\ in^3/min$, how fast is the level of the liquid dropping when the liquid is 5 inches deep?



Solution: Step 1: Related functions / Target functions)

Volume (of the liquid): $V(t)$. Height (level) of the liquid: $h(t)$.

(Key) Known information: $V'(t) = 7$, $h(t) = 5$.

Other Information: size of the filter: 

circular cross-section.

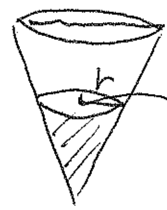
Step 2: (Want to relate V and h in order to find h')

(Search the formula sheet)

- Volume of Cone: $\frac{1}{3} \cdot (\text{height}) \cdot (\text{area of base})$

$$V(t) = \frac{1}{3} \cdot h(t) \cdot \underbrace{(\text{Area of base})}$$

Notice that its circular cross-section

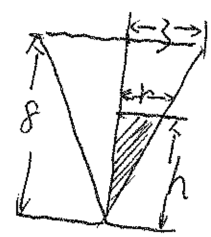


$$= \frac{1}{3} \cdot h(t) \cdot \pi \cdot r^2(t).$$

base is a circle with area: $\pi \cdot r^2$

$r = r(t)$

Now we need to eliminate $r(t)$ via h according to $\underbrace{\text{the geometry of the picture}}$



Similar Triangle:

$$\frac{r}{h} = \frac{3}{8} \Rightarrow r(t) = \frac{3}{8} h(t) \text{ plug in.}$$

$$V(t) = \frac{1}{3} \cdot h(t) \cdot \left[\pi \cdot \left(\frac{3}{8} h(t)\right)^2\right] = \frac{1}{3} h(t) \cdot \pi \cdot \frac{9}{64} [h(t)]^2 = \frac{3\pi}{64} \cdot [h(t)]^3$$

Step 3: Take derivative:

$$V'(t) = \left(\frac{3\pi}{64} h^3(t)\right)' = \frac{3\pi}{64} \cdot \underbrace{3[h(t)]^2}_{\text{out}'(\text{inn})} \cdot \underbrace{h'(t)}_{\text{inn}'}$$

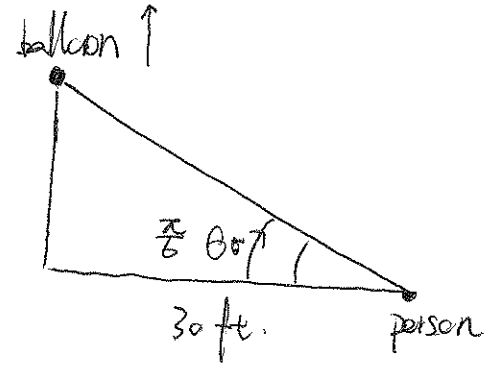
Step 4: Plug in $V' = 7$, $h = 5$.

$$7 = \frac{3\pi}{64} \cdot 3 \cdot 5^2 \cdot h' \xrightarrow[h']{\text{solve for}} h' = \boxed{\frac{7}{\frac{3\pi}{64} \cdot 3 \cdot 25} = \frac{7 \times 64}{9\pi \cdot 25} \quad \text{in/s}}$$

Remark: If we read the problem more carefully, the volume of the liquid is decreasing. Actually, $V' = -7$ and therefore, $h' = -\frac{7 \times 64}{9\pi \cdot 25}$. While the positive answer is also accepted.

eg. 4. A ballon is rising vertically above a field. A person is 30 feet away from the spot on the ground underneath the balloon watching it rise. When the angle of inclination, θ, is $\pi/6$ radians, the angle is increasing at $\frac{1}{10}$ radians per minute. At that moment, how fast is the balloon rising?

(816)



Solution: Step 1:

Target functions:

height of balloon: $h(t)$; angle of inclination: $\theta(t)$.

Known information: $\theta(t) = \frac{\pi}{6}$. $\theta'(t) = \frac{1}{10}$

Step 2: (Relation between h and θ).

$$h(t) = 30 \cdot \tan(\theta(t))$$

Step 3: Take derivative:

$$h'(t) = 30 \cdot [\tan(\theta(t))]' = 30 \cdot \underbrace{\sec^2(\theta(t))}_{out'(inn)} \cdot \underbrace{\theta'(t)}_{inn'}.$$

Remark:
outer: tan ▨
inner: $\theta(t)$

Step 4: Plug in: $\theta = \frac{\pi}{6}$, $\theta' = \frac{1}{10}$.

$$h'(t) = 30 \cdot \sec^2\left(\frac{\pi}{6}\right) \cdot \frac{1}{10}$$

$$= 30 \cdot \left[\sec\left(\frac{\pi}{6}\right)\right]^2 \cdot \frac{1}{10}$$

$$= 30 \cdot \left[\frac{2}{\sqrt{3}}\right]^2 \cdot \frac{1}{10}$$

$$= 30 \cdot \frac{4}{3} \cdot \frac{1}{10} = \boxed{4 \quad ft/min}$$

Remark: $\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$. $\sec = \frac{1}{\cos} = \frac{\text{hypotenuse}}{\text{adjacent}}$.